

GENERALIZED DISTANCE-SQUARED MAPPINGS OF \mathbb{R}^{n+1} INTO \mathbb{R}^{2n+1}

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ABSTRACT. We classify generalized distance-squared mappings of \mathbb{R}^{n+1} into \mathbb{R}^{2n+1} ($n \geq 1$) having generic central points. Moreover, we show that there does not exist a universal bad set $\Sigma \subset (\mathbb{R}^{n+1})^{2n+1}$ in the case of this dimension-pair.

1. INTRODUCTION

For any positive integers k, n , let $p_i = (p_{i0}, p_{i1}, \dots, p_{in})$ ($0 \leq i \leq k$) (resp., $A = (a_{ij})_{0 \leq i \leq k, 0 \leq j \leq n}$) be a point of \mathbb{R}^{n+1} (resp., a $(k+1) \times (n+1)$ matrix with non-zero entries). Let $G_{(p_0, p_1, \dots, p_k, A)} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{k+1}$ be the mapping defined by

$$G_{(p_0, p_1, \dots, p_k, A)}(x) = \left(\sum_{j=0}^n a_{0j}(x_j - p_{0j})^2, \sum_{j=0}^n a_{1j}(x_j - p_{1j})^2, \dots, \sum_{j=0}^n a_{kj}(x_j - p_{kj})^2 \right),$$

where $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$. The mapping $G_{(p_0, p_1, \dots, p_k, A)}$ is called a *generalized distance-squared mapping*. Each component of a generalized distance-squared mapping defines the family of quadrics. The singularities of $G_{(p_0, p_1, \dots, p_k, A)}$ is a helpful information on the contacts of these families. Thus, we may regard that generalized distance-squared mappings are a significant tool in the applications of singularity theory to differential geometry. Therefore, it is natural to classify them. In the case of $n = k = 1$, a recognizable classification is known for $G_{(p_0, p_1, \dots, p_k, A)}$ (see Proposition 1 below). The purpose of this paper is to obtain a classification in the case of $k = 2n$.

Two mappings $f_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{k+1}$ ($i = 1, 2$) are said to be \mathcal{A} -equivalent if there exist C^∞ diffeomorphisms $h : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ and $H : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}$ such that the identity $f_1 = H \circ f_2 \circ h$ holds.

Proposition 1 ([3]). *Let $((x_0, y_0), (x_1, y_1))$ be the standard coordinates of $(\mathbb{R}^2)^2$ and let Σ be the hypersurface in $(\mathbb{R}^2)^2$ defined by $(x_0 - x_1)(y_0 - y_1) = 0$. Let (p_0, p_1) be a point in $(\mathbb{R}^2)^2 - \Sigma$ and let A_k be a 2×2 matrix of rank k with non-zero entries ($k=1, 2$). Then, the following hold:*

- (1) *The mapping $G_{(p_0, p_1, A_1)}$ is proper and stable, and it is \mathcal{A} -equivalent to $(x, y) \mapsto (x, y^2)$.*
- (2) *The mapping $G_{(p_0, p_1, A_2)}$ is proper and stable, and it is not \mathcal{A} -equivalent to $G_{(p_0, p_1, A_1)}$.*
- (3) *Let B_2 be a 2×2 matrix of rank 2 with non-zero entries and let (q_0, q_1) be a point in $(\mathbb{R}^2)^2 - \Sigma$. Then, $G_{(p_0, p_1, A_2)}$ is \mathcal{A} -equivalent to $G_{(q_0, q_1, B_2)}$.*

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In the case of $k = 2n$, only a few partial classification result for $G_{(p_0, \dots, p_k, A)}$ is known. A *distance-squared mapping* $D_{(p_0, p_1, \dots, p_k)}$ (resp., *Lorentzian distance-squared mappings* $L_{(p_0, p_1, \dots, p_k)}$) is the mapping $G_{(p_0, p_1, \dots, p_k, A)}$ satisfying that each entry of A is 1 (resp., $a_{i0} = -1$ and $a_{ij} = 1$ if $j \neq 0$).

Proposition 2 ([1, 2]). *There exists a closed subset $\Sigma \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, p_1, \dots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma$, both $D_{(p_0, p_1, \dots, p_{2n})}$ and $L_{(p_0, p_1, \dots, p_{2n})}$ are \mathcal{A} -equivalent to the inclusion $(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, x_n, 0, \dots, 0)$.*

Theorem 1. *Let $A = (a_{ij})_{0 \leq i \leq 2n, 0 \leq j \leq n}$ be a $(2n+1) \times (n+1)$ matrix with non-zero entries. Then, the following two hold:*

- (1) *Suppose that the rank of A is $n+1$. Then, there exists a closed subset $\Sigma_A \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, p_1, \dots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $G_{(p, A)}$ is \mathcal{A} -equivalent to the following mapping:*

$$(x_0, x_1, \dots, x_n) \mapsto (x_0^2, x_0x_1, \dots, x_0x_n, x_1, \dots, x_n).$$

- (2) *Suppose that the rank of A is less than $n+1$. Then, there exists a closed subset $\Sigma_A \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, p_1, \dots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $G_{(p, A)}$ is \mathcal{A} -equivalent to the inclusion $(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, x_n, 0, \dots, 0)$.*

The mapping given in the assertion (1) of Theorem 1 was firstly given in [5] and is called the *normal form of Whitney umbrella*. It is easily seen that the normal form of Whitney umbrella is not \mathcal{A} -equivalent to the inclusion $(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, x_n, 0, \dots, 0)$. Moreover, by Mather's characterization theorem of stable mappings given in [4], it is easily shown that these two mappings are proper and stable. Thus, Theorem 1 may be regarded as a result of Proposition 1 type. On the other hand, it is desirable to improve Theorem 1 so that the bad set Σ_A given in Theorem 1 does not depend on the given matrix A . However, contrary to the case of $n = k = 1$, in this case it is impossible to expect the existence of such a universal bad set Σ as follows.

Theorem 2. *There does not exist a closed subset $\Sigma \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any point $p = (p_0, p_1, \dots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma$, the following two hold.*

- (1) *Suppose that A is a $(2n+1) \times (n+1)$ matrix with non-zero entries such that the rank of A is $n+1$. Then, $G_{(p, A)}$ is \mathcal{A} -equivalent to the following mapping:*

$$(x_0, x_1, \dots, x_n) \mapsto (x_0^2, x_0x_1, \dots, x_0x_n, x_1, \dots, x_n).$$

- (2) *Suppose that A is a $(2n+1) \times (n+1)$ matrix with non-zero entries such that the rank of A is less than $n+1$. Then, $G_{(p, A)}$ is \mathcal{A} -equivalent to the inclusion $(x_0, x_1, \dots, x_n) \mapsto (x_0, x_1, \dots, x_n, 0, \dots, 0)$.*

The assertion (1) of Theorem 1, the assertion (2) of Theorem 1 and Theorem 2 are proved in Sections 2, 3 and 4 respectively.

2. PROOF OF THE ASSERTION (1) OF THEOREM 1

Set $A_1 = (a_{ij})_{0 \leq i, j \leq n}$ and $A_2 = (a_{ij})_{n+1 \leq i \leq 2n, 0 \leq j \leq n}$. Taking permutations of coordinates of the target space if necessary, without loss of generality, from the first we may assume that $\text{rank}(A_1) = n + 1$.

2.1. STEP 1. The purpose of this step is to delete quadratic terms as many as possible by a linear transformation $H_1 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ of the following type.

$$H_1(X_0, \dots, X_{2n}) = (X_0, \dots, X_{2n}) \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,n} & \lambda_{0,n+1} & \cdots & \lambda_{0,2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{n,0} & \cdots & \lambda_{n,n} & \lambda_{n,n+1} & \cdots & \lambda_{n,2n} \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Set $\Lambda_1 = (\lambda_{i,j})_{0 \leq i, j \leq n}$ and $\Lambda_2 = (\lambda_{i,j})_{0 \leq i \leq n, n+1 \leq j \leq 2n}$. Two matrices Λ_1 and Λ_2 are obtained as the solutions of the following linear equations, where E_{n+1} is the $(n+1) \times (n+1)$ unit matrix and M^T stands for the transposed matrix of a matrix M .

$$A_1^T \Lambda_1 = E_{n+1}, \quad A_1^T \Lambda_2 = -A_2^T.$$

Set $H_1 \circ G_{(p,A)} = (\varphi_{1,0}, \varphi_{1,1}, \dots, \varphi_{1,2n})$. Then, $\varphi_{1,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{1,i}(x_0, x_1, \dots, x_n) = \begin{cases} x_i^2 + \sum_{j=0}^n b_{ij} x_j + c_i & (0 \leq i \leq n) \\ \sum_{j=0}^n b_{ij} x_j + c_i & (n+1 \leq i \leq 2n), \end{cases}$$

where c_i stands for the constant term and b_{ij} is as follows.

$$b_{ij} = \begin{cases} -2 \sum_{k=0}^n \lambda_{k,i} a_{kj} p_{kj} & (0 \leq i \leq n) \\ -2 (\sum_{k=0}^n \lambda_{k,i} a_{kj} p_{kj} + a_{ij} p_{ij}) & (n+1 \leq i \leq 2n). \end{cases}$$

2.2. STEP 2. The purpose of this step is to delete constant terms by the parallel transformation $H_2 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ defined by

$$H_2(X_0, X_1, \dots, X_{2n}) = (X_0 - c_0, X_1 - c_1, \dots, X_{2n} - c_{2n}).$$

Set $H_2 \circ H_1 \circ G_{(p,A)} = (\varphi_{2,0}, \varphi_{2,1}, \dots, \varphi_{2,2n})$. Then, $\varphi_{2,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{2,i}(x_0, x_1, \dots, x_n) = \begin{cases} x_i^2 + \sum_{j=0}^n b_{ij} x_j & (0 \leq i \leq n) \\ \sum_{j=0}^n b_{ij} x_j & (n+1 \leq i \leq 2n), \end{cases}$$

2.3. STEP 3. The purpose of this step is to construct the bad set Σ_A . The desirable bad set Σ_A has the property that for any $p \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $H_2 \circ H_1 \circ G_{(p,A)}$ is transformed to the following type form:

$$(x_0^2 + d_{0,0}x_0, \dots, x_n^2 + d_{n,n}x_n, d_{n+1,0}x_0 + d_{n+1,1}x_1, \dots, d_{2n,0}x_0 + d_{2n,n}x_n).$$

by composing a linear transformation $H_3 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ of the following type.

$$H_3(X_0, X_1, \dots, X_{2n}) = (X_0, X_1, \dots, X_{2n}) \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \gamma_{n+1,0} & \cdots & \gamma_{n+1,n} & \gamma_{n+1,n+1} & \cdots & \gamma_{n+1,2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{2n,0} & \cdots & \gamma_{2n,n} & \gamma_{2n,n+1} & \cdots & \gamma_{2n,2n} \end{pmatrix},$$

where d_{ii} ($0 \leq i \leq n$) are constants and d_{ij} ($n+1 \leq i \leq 2n$) are non-zero constants.

For any j ($0 \leq j \leq n$), set $\mathbf{b}_j = (b_{n+1,j}, \dots, b_{2n,j})^T$. Moreover, for any j ($0 \leq j \leq n$), set

$$B_j = (\mathbf{b}_0, \dots, \mathbf{b}_{j-1}, \hat{\mathbf{b}}_j, \mathbf{b}_{j+1}, \dots, \mathbf{b}_n),$$

where $\hat{\mathbf{b}}_j$ stands for deleting \mathbf{b}_j . Then, B_j is an $n \times n$ matrix for any j ($0 \leq j \leq n$).

Definition 1. (1) For any j ($0 \leq j \leq n$), Σ_{B_j} is the set consisting of points $p \in (\mathbb{R}^{n+1})^{2n+1}$ such that $\det B_j = 0$.

(2)

$$\Sigma_A = \bigcup_{j=0}^n \Sigma_{B_j}.$$

The set Σ_A is closed and of Lebesgue measure zero since it is an algebraic set. Set $H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} = (\varphi_{3,0}, \varphi_{3,1}, \dots, \varphi_{3,2n})$.

Lemma 2.1. Let p be a point of $(\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$. Then, the following hold:

(1) For any j ($0 \leq j \leq n$), $\varphi_{3,j}$ may be expressed as follows where $d_{j,j}$ is a constant.

$$\varphi_{3,j}(x_0, x_1, \dots, x_n) = x_j^2 + d_{j,j}x_j.$$

(2) For any j ($n+1 \leq j \leq 2n$), $\varphi_{3,j}$ may be expressed as follows where $d_{j,0}, d_{j,j-n}$ are non-zero constants.

$$\varphi_{3,j}(x_0, x_1, \dots, x_n) = d_{j,0}x_0 + d_{j,j-n}x_{j-n}.$$

Proof. Since the given point p is outside the bad set Σ_A , it follows that $\det B_j \neq 0$ for any j ($0 \leq j \leq n$). Thus, for any j ($0 \leq j \leq n$), there exists the unique solution $(\gamma_{n+1,j}, \dots, \gamma_{2n,j})^T$ for the following linear equations.

$$(-b_{j,0}, \dots, -b_{j,j-1}, -\hat{b}_{j,j}, -b_{j,j+1}, \dots, -b_{j,n})^T = B_j^T (\gamma_{n+1,j}, \dots, \gamma_{2n,j})^T,$$

where $-\hat{b}_{j,j}$ stands for deleting $-b_{j,j}$. Therefore, the assertion (1) follows.

Next, we show the assertion (2). For any j ($1 \leq j \leq n$), set

$$\tilde{B}_j = (\mathbf{b}_1, \dots, \hat{\mathbf{b}}_j, \dots, \mathbf{b}_n)$$

where $\hat{\mathbf{b}}_j$ stands for deleting \mathbf{b}_j . Then, \tilde{B}_j is an $n \times (n-1)$ matrix for any j ($1 \leq j \leq n$). Since $\det B_j \neq 0$ for any j ($0 \leq j \leq n$), it follows that $\text{rank}(\tilde{B}_j) = n-1$ for any j ($1 \leq j \leq n$). Thus, for any j ($1 \leq j \leq n$), the solution space for the linear equation

$$(0, \dots, 0)^T = \tilde{B}_j^T (\gamma_{n+1,n+j}, \dots, \gamma_{2n,n+j})^T$$

is one-dimensional. Moreover, since $\det B_j \neq 0$ for any j ($0 \leq j \leq n$), both \mathbf{b}_0 and \mathbf{b}_j are not perpendicular to non-zero solution vector $(\gamma_{n+1,n+j}, \dots, \gamma_{2n,n+j})$ for the above linear equations for any j ($1 \leq j \leq n$). Thus, the assertion (2) follows. \square

2.4. STEP 4. The purpose of this step is to complete the square for the first $(n+1)$ components of $H_3 \circ H_2 \circ H_1 \circ G_{(p,A)}$. Set

$$h_1(x_0, x_1, \dots, x_n) = (x_0 - \frac{1}{2}d_{0,0}, x_1 - \frac{1}{2}d_{1,1}, \dots, x_n - \frac{1}{2}d_{n,n})$$

and $H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1 = (\varphi_{4,0}, \varphi_{4,1}, \dots, \varphi_{4,2n})$. Then, $\varphi_{4,i}$ ($0 \leq i \leq 2n$) may be expressed as follows, where $d_{i,0}, d_{i,i-n}$ are non-zero real numbers obtained in STEP 3 and \tilde{d}_i are some constants.

$$\varphi_{4,i}(x_0, x_1, \dots, x_n) = \begin{cases} x_i^2 + \tilde{d}_i & (0 \leq i \leq n) \\ d_{i,0}x_0 + d_{i,i-n}x_{i-n} + \tilde{d}_i & (n+1 \leq i \leq 2n). \end{cases}$$

2.5. STEP 5. The purpose of this step is to delete constant terms by the parallel transformation $H_4 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ defined by

$$H_4(X_0, X_1, \dots, X_{2n}) = (X_0 - \tilde{d}_0, X_1 - \tilde{d}_1, \dots, X_{2n} - \tilde{d}_{2n}).$$

Set $H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1 = (\varphi_{5,0}, \varphi_{5,1}, \dots, \varphi_{5,2n})$. Then, $\varphi_{5,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{5,i}(x_0, x_1, \dots, x_n) = \begin{cases} x_i^2 & (0 \leq i \leq n) \\ d_{i,0}x_0 + d_{i,i-n}x_{i-n} & (n+1 \leq i \leq 2n). \end{cases}$$

2.6. STEP 6. The purpose of this step is to simplify the last n components of $H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1$. Set

$$\tilde{x}_0 = x_0, \tilde{x}_1 = d_{n+1,0}x_0 + d_{n+1,1}x_1, \dots, \tilde{x}_n = d_{i,0}x_0 + d_{2n,n}x_n.$$

Then, since $d_{i,i-n} \neq 0$ for any i ($n+1 \leq i \leq 2n$), the mapping $(x_0, x_1, \dots, x_n) \mapsto (\tilde{x}_0, \tilde{x}_1, \dots, \tilde{x}_n)$ is a linear transformation of \mathbb{R}^{n+1} . Thus, setting

$$h_2(x_0, x_1, \dots, x_n) = \left(x_0, \frac{1}{d_{n+1,1}}(x_1 - d_{n+1,0}x_0), \dots, \frac{1}{d_{2n,n}}(x_n - d_{2n,0}x_0) \right)$$

and composing h_2 to $H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1$, the desired form may be obtained as follows:

$$\begin{aligned} & H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1 \circ h_2(x_0, x_1, \dots, x_n) \\ &= \left(x_0^2, \frac{1}{d_{n+1,1}^2}(x_1 - d_{n+1,0}x_0)^2, \dots, \frac{1}{d_{2n,n}^2}(x_n - d_{2n,0}x_0)^2, x_1, \dots, x_n \right). \end{aligned}$$

2.7. STEP 7. This is the final step. Notice that $d_{i,0} \neq 0$ for any i ($n+1 \leq i \leq 2n$). Thus, the following coordinate transformation H_5 is well-defined.

$$\begin{aligned} & H_5(X_0, X_1, \dots, X_{2n}) \\ &= \left(X_0, -\frac{d_{n+1,1}^2}{2d_{n+1,0}} \left(X_1 - \frac{d_{n+1,0}^2}{d_{n+1,1}^2} X_0 - \frac{1}{d_{n+1,1}^2} X_{n+1}^2 \right), \dots, \right. \\ & \quad \left. -\frac{d_{2n,n}^2}{2d_{2n,0}} \left(X_n - \frac{d_{2n,0}^2}{d_{2n,n}^2} X_0 - \frac{1}{d_{2n,n}^2} X_{2n}^2 \right), X_{n+1}, \dots, X_{2n} \right). \end{aligned}$$

Then, the desired normal form of Whitney umbrella may be obtained as follows:

$$\begin{aligned} & H_5 \circ H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} \circ h_1 \circ h_2(x_0, x_1, \dots, x_n) \\ &= (x_0^2, x_0 x_1, \dots, x_0 x_n, x_1, \dots, x_n). \end{aligned}$$

□

3. PROOF OF THE ASSERTION (2) OF THEOREM 1

Set $A_3 = (a_{ij})_{0 \leq i, j \leq n-1}$. Taking permutations of coordinates of the target space if necessary, without loss of generality, from the first we may assume that $\text{rank}(A_3) = \text{rank}(A) \leq n$.

3.1. STEP 1. The purpose of this step is to delete quadratic terms as many as possible by a linear transformation $H_1 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ of the following type.

$$\begin{aligned} & H_1(X_0, X_1, \dots, X_{2n}) \\ &= (X_0, X_1, \dots, X_{2n}) \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \lambda_{0,n} & \cdots & \lambda_{0,2n} \\ 0 & 1 & \ddots & 0 & 0 & \lambda_{1,n} & \cdots & \lambda_{1,2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \lambda_{n-2,n} & \cdots & \lambda_{n-2,2n} \\ 0 & 0 & \cdots & 0 & 1 & \lambda_{n-1,n} & \cdots & \lambda_{n-1,2n} \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

For any i ($1 \leq i \leq 2n+1$), let \mathbf{a}_{i-1} be the i -th row vector of A . Then, by the assumption of $\text{rank}(A_3) = \text{rank}(A) \leq n$, the following claim holds:

Claim 3.1. *For any i ($n \leq i \leq 2n+1$), there exist $\alpha_{0i}, \dots, \alpha_{(n-1)i} \in \mathbb{R}$ such that the following equality holds, where $\mathbf{0}$ is the $(n+1)$ dimensional zero row vector:*

$$\sum_{j=0}^{n-1} \alpha_{ji} \mathbf{a}_j + \mathbf{a}_i = \mathbf{0}.$$

Set $H_1 \circ G_{(p,A)} = (\varphi_{1,0}, \varphi_{1,1}, \dots, \varphi_{1,2n})$. Then, $\varphi_{1,i}$ ($0 \leq i \leq 2n$) may be expressed as follows:

$$\varphi_{1,i}(x_0, x_1, \dots, x_n) = \begin{cases} \sum_{j=0}^n a_{ij}(x_j - p_{ij})^2 & (0 \leq i \leq n-1) \\ \sum_{j=0}^n b_{ij}x_j + c_i & (n \leq i \leq 2n), \end{cases}$$

where b_{ij}, c_i stands for some constants.

3.2. STEP 2. The purpose of this step is to delete constant terms c_i by the parallel transformation $H_2 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ defined by

$$H_2(X_0, X_1, \dots, X_{2n}) = (X_0, X_1, \dots, X_{n-1}, X_n - c_n, \dots, X_{2n} - c_{2n}).$$

Set $H_2 \circ H_1 \circ G_{(p,A)} = (\varphi_{2,0}, \varphi_{2,1}, \dots, \varphi_{2,2n})$. Then, $\varphi_{2,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{2,i}(x_0, x_1, \dots, x_n) = \begin{cases} \sum_{j=0}^n a_{ij}(x_j - p_{ij})^2 & (0 \leq i \leq n-1) \\ \sum_{j=0}^n b_{ij}x_j & (n \leq i \leq 2n), \end{cases}$$

3.3. STEP 3. The purpose of this step is to construct the bad set Σ_A . Set $B = (b_{ij})_{n \leq i \leq 2n, 0 \leq j \leq n}$. Then, B is an $(n+1) \times (n+1)$ matrix.

Definition 2. Σ_A is the set consisting of points $p \in (\mathbb{R}^{n+1})^{2n+1}$ such that $\det B = 0$.

The set Σ_A is closed and of Lebesgue measure zero since it is an algebraic set. Take a point $p \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$. Then, by the construction of Σ_A , the matrix B has its inverse matrix B^{-1} . Let $H_3 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ be the linear transformation defined as follows:

$$\begin{aligned} & H_3(X_0, X_1, \dots, X_{2n}) \\ &= (X_0, X_1, \dots, X_{2n}) \left(\frac{E_n}{0} \middle| \frac{0}{(B^T)^{-1}} \right), \end{aligned}$$

where E_n is the $n \times n$ unit matrix.

Set $H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} = (\varphi_{3,0}, \varphi_{3,1}, \dots, \varphi_{3,2n})$. Then, $\varphi_{3,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{3,i}(x_0, x_1, \dots, x_n) = \begin{cases} \sum_{j=0}^n a_{ij}(x_j - p_{ij})^2 & (0 \leq i \leq n-1) \\ x_{i-n} & (n \leq i \leq 2n). \end{cases}$$

3.4. STEP 4. The purpose of this step is to delete remaining constant terms by the parallel transformation $H_4 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ defined by

$$\begin{aligned} & H_4(X_0, X_1, \dots, X_{2n}) \\ &= (X_0 - \sum_{j=0}^n a_{0j} p_{0j}^2, X_1 - \sum_{j=0}^n a_{1j} p_{1j}^2, \dots, X_{n-1} - \sum_{j=0}^n a_{(n-1)j} p_{(n-1)j}^2, \\ & \quad X_n, X_{n+1}, \dots, X_{2n}). \end{aligned}$$

Set $H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)} = (\varphi_{4,0}, \varphi_{4,1}, \dots, \varphi_{4,2n})$. Then, $\varphi_{4,i}$ ($0 \leq i \leq 2n$) may be expressed as follows.

$$\varphi_{4,i}(x_0, x_1, \dots, x_n) = \begin{cases} \sum_{j=0}^n a_{ij}(x_j^2 - 2p_{ij}x_j) & (0 \leq i \leq n-1) \\ x_{i-n} & (n \leq i \leq 2n). \end{cases}$$

3.5. STEP 5. This is the final step. Let $H_5 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ be the coordinate transformation defined by

$$\begin{aligned} & H_5(X_0, X_1, \dots, X_{2n}) \\ &= \left(X_n, X_{n+1}, \dots, X_{2n}, X_0 - \sum_{j=0}^n a_{0j} (X_{n+j}^2 - 2p_{0j}X_{n+j}), \right. \\ & \quad X_1 - \sum_{j=0}^n a_{1j} (X_{n+j}^2 - 2p_{1j}X_{n+j}), \dots, \\ & \quad \left. X_{n-1} - \sum_{j=0}^n a_{(n-1)j} (X_{n+j}^2 - 2p_{(n-1)j}X_{n+j}) \right). \end{aligned}$$

Then, we have the following:

$$H_5 \circ H_4 \circ H_3 \circ H_2 \circ H_1 \circ G_{(p,A)}(x_0, x_1, \dots, x_n) = (x_0, \dots, x_n, 0, \dots, 0).$$

□

3.6. REMARK. As a by-product of the proof of the assertion (2) of Theorem 1 given in this section, we have the following:

Theorem 3. *Let n be a positive integer and k be an integer such that $k > 2n$. Let $A = (a_{ij})_{0 \leq i \leq k, 0 \leq j \leq n}$ be a $(k+1) \times (n+1)$ matrix with non-zero entries. Then, there exists a closed subset $\Sigma_A \subset (\mathbb{R}^{n+1})^{2n+1}$ with Lebesgue measure zero such that for any $p = (p_0, \dots, p_{2n}) \in (\mathbb{R}^{n+1})^{2n+1} - \Sigma_A$, $G_{(p,A)}$ is \mathcal{A} -equivalent to the inclusion $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0, \dots, 0)$.*

4. PROOF OF THEOREM 2

It is sufficient to show the following proposition.

Proposition 3. *Let X be the set consisting of $(2n+1) \times (n+1)$ matrices A with non-zero entries such that the rank of A is $n+1$. Then, for any $A \in X$ there exist an open neighborhood U_A of A in X and an open set $V \subset (\mathbb{R}^{n+1})^{2n+1}$ such that for any $p \in V$ there exists $B \in U_A$ satisfying that $G_{(p,B)}$ is an unstable mapping.*

Proof of Proposition 3. Let $A = (a_{ij})_{0 \leq i \leq 2n, 0 \leq j \leq n}$ be a given element of X . Taking a permutation of coordinates of the target space if necessary, without loss of generality, from the first we may assume that the rank of A_1 is $n+1$, where $A_1 = (a_{ij})_{0 \leq i, j \leq n}$ as defined in the beginning of Section 2.

Let $\tilde{A} = (\tilde{a}_{ij})_{0 \leq i \leq 2n, 0 \leq j \leq n} \in X$ be a sufficiently near matrix to A such that $\tilde{a}_{ij} = a_{ij}$ if $0 \leq i, j \leq n$. Set $\tilde{A}_2 = (\tilde{a}_{ij})_{n+1 \leq i \leq 2n, 0 \leq j \leq n}$. For the matrix \tilde{A} , consider the matrix $\Lambda_2 = (\lambda_{i,j})_{0 \leq i \leq n, n+1 \leq j \leq 2n}$ defined in STEP 1 of Section 2:

$$\Lambda_2 = -(A_1^T)^{-1} \tilde{A}_2^T.$$

Moreover, as in Section 2, for any i, j ($n+1 \leq i \leq 2n, 0 \leq j \leq n$), consider b_{ij} for the matrix \tilde{A} .

$$b_{ij} = -2 \left(\sum_{k=0}^n \lambda_{k,i} a_{kj} p_{kj} + \tilde{a}_{ij} p_{ij} \right),$$

where the real number $\lambda_{k,i}$ is the $(k, i-n)$ component of Λ_2 . Notice that $\lambda_{k,i}$ in b_{ij} is a linear function with respect to \tilde{a}_{ij} ($n+1 \leq i \leq 2n, 0 \leq j \leq n$). Thus, for any i_0, j_0 ($n+1 \leq i_0 \leq 2n, 0 \leq j_0 \leq n$), the following function $\tilde{\psi}_{i_0, j_0}$ is a rational function with variables $p_{00}, \dots, p_{nn}, \tilde{a}_{(n+1)0}, \dots, \tilde{a}_{(2n)n}$.

$$\tilde{\psi}_{i_0, j_0}(p_{00}, \dots, p_{nn}, \tilde{a}_{(n+1)0}, \dots, \tilde{a}_{(2n)n}) = \frac{-\sum_{k=0}^n \lambda_{k, i_0} a_{kj_0} p_{kj_0}}{\tilde{a}_{i_0 j_0}}.$$

We would like to show that there exist a matrix \tilde{A} of the above type which is sufficiently near A and an open set V in $(\mathbb{R}^{n+1})^{2n+1}$ such that $b_{ij} = 0$ for any point $p \in V$. In order to do so, we consider the mapping

$$\Psi = (\psi_{i_0, j_0})_{0 \leq i_0 \leq 2n, 0 \leq j_0 \leq n} : \mathbb{R}^{(n+1)^2} \times (\mathbb{R} - \{\mathbf{0}\})^{n(n+1)} \rightarrow \mathbb{R}^{(2n+1)(n+1)}$$

defined as follows:

$$\begin{aligned} & \psi_{i_0, j_0}(q_{00}, \dots, q_{nn}, c_{(n+1)0}, \dots, c_{(2n)n}) \\ &= \begin{cases} q_{i_0 j_0} & (0 \leq i_0 \leq n) \\ \psi_{i_0, j_0}(q, c) & (n+1 \leq i_0 \leq 2n), \end{cases} \end{aligned}$$

where $q = (q_{00}, \dots, q_{nn})$ and $c = (c_{(n+1)0}, \dots, c_{(2n)n})$.

Definition 3.

$$\tilde{\Sigma} = \left\{ (q, c) \in \mathbb{R}^{(n+1)^2} \times (\mathbb{R} - \{\mathbf{0}\})^{n(n+1)} \mid \det J\Psi(q, c) = 0 \right\},$$

where $J\Psi(q, c)$ is the Jacobian matrix of Ψ at (q, c) .

Since ψ_{i_0, j_0} is a rational function, $\tilde{\Sigma}$ is a semi-algebraic subset of Lebesgue measure zero. Let (q_0, c_0) be a point outside $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is closed and of Lebesgue measure zero, we may assume that c_0 is sufficiently near A_2 . Since $\det J\Psi(q_0, c_0) \neq 0$, by the inverse function theorem, there exist open neighborhoods $U_1 \subset \mathbb{R}^{(n+1)^2}$ of q_0 , $U_2 \subset (\mathbb{R} - \{\mathbf{0}\})^{n(n+1)}$ of c_0 and $V \subset \mathbb{R}^{(2n+1)(n+1)}$ of $\Psi(q_0, c_0)$ such that the restriction $\Psi|_{U_1 \times U_2} : U_1 \times U_2 \rightarrow V$ is a C^∞ diffeomorphism. In particular, we have the following:

Claim 4.1. *For any $p = (p_{00}, \dots, p_{(2n)_n}) \in V$ there exists a matrix $\tilde{A}_2 \in U_2$ such that*

$$p_{i_0 j_0} = \tilde{\psi}_{i_0, j_0}(p_{00}, \dots, p_{nn}, \tilde{A}_2)$$

for any i_0, j_0 ($n+1 \leq i_0 \leq 2n, 0 \leq j_0 \leq n$).

Claim 4.1 implies the following:

Claim 4.2. *For any $p = (p_{00}, \dots, p_{(2n)_n}) \in V$ there exists a matrix $\tilde{A}_2 \in U_2$ such that $b_{i_0 j_0} = 0$ for any i_0, j_0 ($n+1 \leq i_0 \leq 2n, 0 \leq j_0 \leq n$).*

Claim 4.2 shows that the image of $H_1 \circ G_{(p, \tilde{A})}$ must be inside $\mathbb{R}^{n+1} \times \{\mathbf{0}\}$ where H_1 is the linear transformation given in STEP 1 of Section 2. Thus, by the classification of stable singularity for map-germ $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n+1}$ due to Whitney ([5]), if $G_{(p, \tilde{A})}$ is stable, then it must be \mathcal{A} -equivalent to the inclusion $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0, \dots, 0)$. In particular, $G_{(p, \tilde{A})}$ must be non-singular. However, it is easily seen that $G_{(p, \tilde{A})}$ is singular. Hence, $G_{(p, \tilde{A})}$ is unstable. \square

4.1. REMARK.

- (1) As a by-product of the proof of Theorem 2 given in this section, we have the following:

Theorem 4. *Let n be a positive integer and k be an integer such that $k > 2n$. Then, there does not exist a semi-algebraic subset $\Sigma \subset (\mathbb{R}^{n+1})^{k+1}$ with Lebesgue measure zero such that for any point $p = (p_0, p_1, \dots, p_k) \in (\mathbb{R}^{n+1})^{k+1} - \Sigma$ and any $(k+1) \times (n+1)$ matrix A with non-zero entries, $G_{(p, A)}$ is \mathcal{A} -equivalent to the inclusion $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0, \dots, 0)$.*

- (2) The proof of Theorem 2 has one more advantage. It makes clear the reason why we can expect the existence of a universal bad set Σ in the case $n = k = 1$.

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